ALMOST KÄHLER STRUCTURES ON FOUR DIMENSIONAL UNIMODULAR LIE ALGEBRAS

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Abstract. Let J be an almost complex structure on a 4-dimensional and unimodular Lie algebra $\mathfrak{g}$. We show that there exists a symplectic form taming J if and only if there is a symplectic form compatible with J. We also introduce groups $H^+_J(\mathfrak{g})$ and $H^-_J(\mathfrak{g})$ as the subgroups of the Chevalley-Eilenberg cohomology classes which can be represented by J-invariant, respectively J-anti-invariant, 2-forms on $\mathfrak{g}$. and we prove a cohomological $J-$decomposition theorem following [9]: $H^2(\mathfrak{g}) = H^+_J(\mathfrak{g}) \oplus H^-_J(\mathfrak{g})$. We discover that tameness of J can be characterized in terms of the dimension of $H^\pm_J(\mathfrak{g})$, just as in the complex surface case. We also describe the tamed and compatible symplectic cones respectively. Finally, two applications to homogeneous J on $4-$manifolds are obtained.

Contents

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Introduction

In this paper we are interested in the geometry of almost complex structures on four dimensional Lie algebras and their cohomological properties, especially from the symplectic point of view.

Given a $2n$-dimensional oriented real Lie algebra $\mathfrak{g}$, let $\Lambda^k(\mathfrak{g})$ and $\Lambda^r(\mathfrak{g}^*)$ denote the spaces of k-vectors, r-forms respectively on $\mathfrak{g}$. Let $H^*(\mathfrak{g})$ be the Chevalley-Eilenberg cohomology group of $\mathfrak{g}$. Then $\mathfrak{g}$ is called *unimodular* if its top dimensional cohomology vanishes.

The special feature in dimension four are, up to a choice of a volume form, the symmetric wedge product pairing on $\Lambda^2(\mathfrak{g}^*)$ (as well as the one on $\Lambda^2(\mathfrak{g})$) with signature $(3,3)$, and the induced one on $H^2(\mathfrak{g})$ with signature $(b^+(\mathfrak{g}), b^+(\mathfrak{g}))$ when $\mathfrak{g}$ is unimodular. Here $b^+(\mathfrak{g})$ is the maximal dimension of a definite subspace of $H^2(\mathfrak{g})$.

Let J be an almost complex structure on $\mathfrak{g}$, i.e. an automorphism of $\mathfrak{g}$ with $J^2 = -id$, such that the natural orientation induced by J is the same as the fixed one.
Consider the involution on the space of 2-forms $\Lambda^2(\mathfrak{g}^*)$, $\alpha(\cdot,\cdot) \to \alpha(J\cdot, J\cdot)$. Accordingly, we have the $\pm 1-$eigenspace decomposition:

$$\Lambda^2(\mathfrak{g}^*) = \Lambda^+_J(\mathfrak{g}^*) \oplus \Lambda^-_J(\mathfrak{g}^*). \tag{1}$$

We recall the following

Definition 0.1. *Let $\mathfrak{g}$ be endowed with an almost complex structure J. A closed 2-form φ on $\mathfrak{g}$ is said to be $J-$*tamed *if it is positive on $v \wedge Jv$ for any nonzero $v \in \mathfrak{g}$. A $J-$tamed φ is said to be $J-$*compatible *if $\varphi \in \Lambda^+_J(\mathfrak{g}^*)$. The almost complex structure J is said to be* tamed *if there exists a $J-$tamed form and* almost Kähler*, in short,* AK*, if there exists a $J-$compatible form.*

Our first main result is (see Theorem 2.5):

Theorem 0.2. *Let $\mathfrak{g}$ be a $4-$dimensional Lie algebra with $B \wedge B = 0$, where $B \subset \Lambda^2(\mathfrak{g})$ is the space of boundary $2-$vectors. Then an almost complex structure J is tamed if and only if J is almost Kähler.*

If $\mathfrak{g}$ is $4-$dimensional and unimodular, then the condition $B \wedge B = 0$ is satisfied. In particular, Theorem 0.2 confirms the tame/compatible Question of Donaldson (see [8, Question 2]) in the Lie algebra setting.

We further discover that tamed almost complex structures can be characterized cohomologically. For this purpose, introduce the following subgroups $H^\pm_J(\mathfrak{g})$ of $H^2(\mathfrak{g})$.

Definition 0.3. *Let $\mathcal{Z}^2$ denote the space of closed 2-forms on $\mathfrak{g}$ and let $\mathcal{Z}^\pm_J = \mathcal{Z}^2 \cap \Lambda^\pm_J(\mathfrak{g}^*)$. Define*

$$H^\pm_J(\mathfrak{g}) = \{\mathfrak{a} \in H^2(\mathfrak{g}) \mid \exists\, \alpha \in \mathcal{Z}^\pm_J \text{ such that } [\alpha] = \mathfrak{a}\}. \tag{2}$$

If $\mathfrak{g}$ is 4-dimensional and unimodular, we establish a $J-$decomposition result for $H^2(\mathfrak{g})$, i.e.

$$H^2(\mathfrak{g}) = H^+_J(\mathfrak{g}) \oplus H^-_J(\mathfrak{g})$$

(see Theorem 3.3).

The following is our second main result, which is a characterization of tamed almost complex structures in terms of the dimension h^- of $H^-_J(\mathfrak{g})$ (see Theorem 3.7):

Theorem 0.4. *Suppose J is an almost complex structure on a 4-dimensional unimodular Lie algebra $\mathfrak{g}$. Then $h_J^- = b^+(\mathfrak{g})$ or $b^+(\mathfrak{g}) - 1$. Moreover, J is tamed if and only if*

$$h_J^- = b^+(\mathfrak{g}) - 1.$$

A particular interesting consequence (see Theorem 3.7) is that, when $b^+(\mathfrak{g}) = 2$, an almost complex structure is either integrable or almost Kähler.

We also introduce tamed and compatible cones, and we are able to describe both of them when the Lie algebra is 4−dimensional and unimodular.

Definition 0.5. *The convex cones*

$$\mathcal{K}_J^t = \left\{[\omega] \in H^2(\mathfrak{g}) \;\mid\; \omega \text{ is } J\text{-tamed}\right\} \qquad \textit{and} \quad \mathcal{K}_J^c = \left\{[\omega] \in H_J^+(\mathfrak{g}) \;\mid\; \omega \text{ is } J\text{-compatible}\right\}$$

*are called the $J-$*tamed symplectic cone *and the $J-$*compatible symplectic cone *respectively.*

We show the following (see Theorem 3.10)

Theorem 0.6. *For an almost complex structure J on a 4-dimensional unimodular Lie algebra, the $J-$compatible cone $\mathcal{K}_J^c(\mathfrak{g})$ is a connected component of $\{e \in H_J^+(\mathfrak{g}) | e^2 > 0\}$, and the $J-$tamed cone $\mathcal{K}_J^t(\mathfrak{g}) = \mathcal{K}_J^c(\mathfrak{g}) + H_J^-(\mathfrak{g})$.*

We want to emphasize that although there is a classification of unimodular symplectic 4-dimensional Lie algebras ([27]), our proofs of Theorems 0.4 and 0.6 do not rely on it. We systematically explore the pairings on Λ^2, and it seems that much can be extended to certain class of higher dimensional unimodular symplectic Lie algebras.

We finally note that the results above obtained at the Lie algebra level have analogues for a left-invariant almost complex structure J on a quotient $M = \Gamma\backslash G$ of a Lie group G by a discrete subgroup $\Gamma \subset G$.

The paper is organized as follows: in Section 1 we start by describing carefully the natural non-degenerate bilinear pairings on $\Lambda^k(\mathfrak{g})$ and their properties and by recalling the basic definitions of almost Hermitian geometry on Lie algebras. Sections 2 and 3 are devoted to the proofs of the main results, namely Theorems 0.2, 0.4 and 0.6, which are included in Theorems 2.5, 3.7 and 3.10 respectively. Key tools in their proofs are a finite dimensional version of the Hahn-Banach theorem together with the results by [29, Theorem 3.2] and [17, Theorem 14]. In particular, in Section 2 we prove Theorem 2.5. In Section 3, we focus on unimodular 4-dimensional Lie algebras endowed with an almost complex structure J. We prove Theorem 3.3 and, as a consequence, we observe that there is a homological $J-$decomposition on $\mathfrak{g}$ (see Remark 3.5). As an application of the Theorem 0.4, we construct a 2-parameter family of almost complex structures on the Lie algebras $\mathfrak{nil}^3 \times \mathbb{R}$ and $\mathfrak{nil}^4$ (see subsection 3.5) which cannot be tamed by any symplectic form. In Section 4 we establish the analogous results for homogeneous J on closed 4−manifolds.

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1. Almost complex geometry on Lie algebras

We start by presenting some preliminaries and fixing some notations.

Let $\mathfrak{g}$ be a $2n$-dimensional real Lie algebra, which we will assume to be oriented. The Lie bracket induces the operator $d : \Lambda^1(\mathfrak{g}^*) \to \Lambda^2(\mathfrak{g}^*)$ by $d\phi(u, v) = -\phi([u, v])$, which makes $\Lambda^*(\mathfrak{g}^*)$ a differential algebra by the Leibniz rule. This is the so called *Chevalley-Eilenberg complex* of the Lie algebra $\mathfrak{g}$.

We will denote by $\mathcal{Z}^r$, $\mathcal{B}^r$, the space of closed, exact $r-$forms respectively on $\mathfrak{g}$. Then the $r-$th Chevalley-Eilenberg cohomology is $H^r(\mathfrak{g}) = \mathcal{Z}^r/\mathcal{B}^r$.

Let us also define the homology $H_*(\mathfrak{g})$ of $\mathfrak{g}$. Given any k-vector $u \in \Lambda^k(\mathfrak{g})$, set

$$T_u(\varphi) = \varphi(u),$$

for every $\varphi \in \Lambda^k(\mathfrak{g}^*)$. Then T_u is a linear functional on $\Lambda^k(\mathfrak{g}^*)$. For every $w \in \Lambda^k(\mathfrak{g})$, set

$$dT_w(\varphi) = (-1)^{k+1}T_w(d\varphi),$$

for every $\varphi \in \Lambda^{k-1}(\mathfrak{g}^*)$. Then, dT_w is a linear functional on $\Lambda^{k-1}(\mathfrak{g}^*)$. A k-vector u is a *k-cycle* if $dT_u = 0$, a *k-boundary* if there exists a $(k+1)$-vector w such that $T_u = dT_w$. Now the $k-$th homology of $\mathfrak{g}$ is the quotient of the space of $k-$cycles by the space of $k-$boundaries. In the sequel, we will denote by Z^k, B^k, the space of k-cycles, k-boundaries on $\mathfrak{g}$ respectively.

1.1. **Bilinear pairings.** We will consider a number of non-degenerate bilinear pairings including

$$\Psi^k : \Lambda^k(\mathfrak{g}) \times \Lambda^k(\mathfrak{g}^*) \to \mathbb{R}, \quad \Psi^k(u, \alpha) = \alpha(u),$$

for each k. Let us first state some general properties of a non-degenerate bilinear pairing on finite dimensional vector spaces.

Let V, W be real vector spaces and $\Gamma : V \times W \to \mathbb{R}$ a bilinear pairing. Let $V_0 \subset V$ and $W_0 \subset W$ be subspaces and set

$$V_0^\perp = \{w \in W \mid \Gamma(v, w) = 0, \text{ for any } v \in V_0\},$$

$$W_0^\perp = \{v \in V \mid \Gamma(v, w) = 0, \text{ for any } w \in W_0\}.$$

We say that $V_0 \subset V$ and $W_0 \subset W$ are a $\Gamma-$*complementary pair*, if

$$V_0 = W_0^\perp \quad \text{and} \quad W_0 = V_0^\perp.$$

Lemma 1.1. *Suppose V and W have finite dimension and $\Gamma : V \times W \to \mathbb{R}$ is a non-degenerate pairing. Then*

- $\dim V = \dim W$*;*
- $\dim W_0 + \dim W_0^\perp = \dim W$ *for any $W_0 \subset W$;*
- *For any $V_0 \subset V$ and $W_0 \subset W$, if $V_0 = W_0^\perp$ then $W_0 = V_0^\perp$;*
- *If $V_1 \subset V_0$, $W_0 \subset W_1$, (V_0, W_0) and (V_1, W_1) are both $\Gamma-$complementary pair, then there is an induced pairing $\bar{\Gamma} : V_0/V_1 \times W_1/W_0 \to \mathbb{R}$, which is also non-degenerate.*

This is straightforward since $\dim V$ and $\dim W$ are finite.

Lemma 1.2. *Suppose V and W have finite dimension and $\Gamma : V \times W \to \mathbb{R}$ is a non-degenerate pairing. Suppose that there are involutions ι_V, ι_W on V and W such that*

$$\Gamma(\iota_V(\cdot), \iota_W(\cdot)) = \Gamma(\cdot, \cdot). \tag{3}$$

Let $V^{\pm}$ and $W^{\pm}$ be the $\pm-$eigenspaces of ι_V and ι_W respectively, and

$$\Gamma_{\pm} : V^{\pm} \times W^{\pm} \to \mathbb{R}$$

be the restriction of Γ. Then

- $\Gamma(V^{\pm}, W^{\mp}) = 0$;
- *Furthermore, $V^{\pm}$ and $W^{\mp}$ are a $\Gamma-$complementary pair;*
- *$\Gamma_{\pm}$ is non-degenerate.*

Proof. Suppose $v \in V^{\pm}$ and $w \in W^{\mp}$. Then by (3),

$$\Gamma(v, w) = \Gamma(\iota_V(v), \iota_W(w)) = -\Gamma(v, w).$$

Thus $\Gamma(v, w) = 0$.

For the second bullet we just show that V^+ and W^- are a $\Gamma-$complementary pair, the other case is similar. Since Γ is non-degenerate, by the third bullet of Lemma 1.1 it is sufficient to prove that $V^+ = (W^-)^{\perp}$. By the first bullet we have $V^+ \subset (W^-)^{\perp}$.

Conversely, let $v \in (W^-)^{\perp}$. Write $v = v_+ + v_-$, where $v_{\pm} \in V^{\pm}$. Given any $w \in W^-$, we have:

$$0 = \Gamma(v, w) = \Gamma(v_+ + v_-, w) = \Gamma(v_-, w).$$

Therefore, $\Gamma(v_-, W^+ \oplus W^-) = 0$. Consequently, $v_- = 0$ since Γ is non-degenerate. Hence $v = v_+ \in V^+$ and $V^+ \supset (W^-)^{\perp}$.

The third bullet is a direct consequence of the second bullet and the non-degeneracy of Γ. □

Applying Lemma 1.1 we obtain

Lemma 1.3. *$(Z_k, \mathcal{B}^k)$ is a Ψ^k-complementary pair, and so is $(B_k, \mathcal{Z}^k)$. In particular, there is an induced pairing*

$$\bar{\Psi}^k : H_k(\mathfrak{g}) \times H^k(\mathfrak{g}) \to \mathbb{R},$$

which is also non-degenerate.

Proof. By the fourth bullet of Lemma 1.1, the last statement follows from the first statement. For the first statement we just treat the first pair, the second pair is similar. Further, by the third bullet of Lemma 1.1 we just need to show that $Z_k = (\mathcal{B}^k)^{\perp}$.

($\subset$) Let $u \in Z_k$. Then, for every $\gamma \in \Lambda^{k-1}(\mathfrak{g}^*)$,

$$\Psi^k(u, d\gamma) = T_u(d\gamma) = (-1)^{k+1} dT_u(\gamma) = 0,$$

i.e., $u \in (\mathcal{B}^k)^{\perp}$.

($\supset$) Let $u \in (\mathcal{B}^k)^{\perp}$. Then,

$$0 = \Psi^k(u, d\gamma) = T_u(d\gamma) = (-1)^{k+1} dT_u(\gamma)$$

for every $\gamma \in \Lambda^{k-1}(\mathfrak{g}^*)$. Hence, $dT_u = 0$ and $u \in Z_k$.

□

We also need to consider the following pairings. Fix $\zeta \neq 0 \in \Lambda^{2n}(\mathfrak{g})$ inducing the given orientation of $\mathfrak{g}$, and $\eta \in \Lambda^{2n}(\mathfrak{g}^*)$ with $\Psi^{2n}(\zeta, \eta) = 1$.

Definition 1.4. *For each k, we define the bilinear pairing*

$$\Phi_\zeta^k : \Lambda^k(\mathfrak{g}^*) \times \Lambda^{2n-k}(\mathfrak{g}^*) \to \mathbb{R},$$

$$\Phi_\zeta^k(\alpha, \beta) = (\alpha \wedge \beta)(\zeta), \tag{4}$$

for every $\alpha \in \Lambda^k(\mathfrak{g}^*)$, $\beta \in \Lambda^{2n-k}(\mathfrak{g}^*)$, and similarly define

$$\Phi_\eta^k : \Lambda^k(\mathfrak{g}) \times \Lambda^{2n-k}(\mathfrak{g}) \to \mathbb{R}, \qquad \Phi_\eta^k(u, w) = \eta(u \wedge w).$$

A $k-$vector in $\Lambda^k(\mathfrak{g})$ is called *simple* if it is of the form $w_1 \wedge \ldots \wedge w_k$ for $w_i \in \mathfrak{g}$. *Simple forms* are defined similarly. Geometrically, a non-zero simple $k-$vector $w_1 \wedge \ldots \wedge w_k$ gives rise to a $k-$plane, $\mathrm{Span}\{w_1, \ldots, w_k\}$. In fact, $v \in \mathfrak{g}$ belongs to $\mathrm{Span}\{w_1, \ldots, w_k\}$ if and only if $v \wedge (w_1 \wedge \ldots \wedge w_k) = 0$. If $\alpha \in \Lambda^k(\mathfrak{g}^*)$ and $\{v_1, \ldots, v_k\} \subset \mathfrak{g}$, our convention is that $\alpha(v_1 \wedge \cdots \wedge v_k) = \alpha(v_1, \ldots, v_k)$.

If we choose a basis $\{w_i\}$ of $\mathfrak{g}$ and the dual basis $\{w^i\}$ of $\mathfrak{g}^*$, for each k, there are associated bases of $\Lambda^k(\mathfrak{g})$ and $\Lambda^k(\mathfrak{g}^*)$ consisting of simple $k-$vectors and $k-$forms. By considering such bases, it is easy to prove

Lemma 1.5. *1. Φ_ζ^k and Φ_η^k are non-degenerate pairings for each k.*

2. The linear maps

$$G_\eta^k : \Lambda^k(\mathfrak{g}) \to \Lambda^{2n-k}(\mathfrak{g}^*), \quad G_\zeta^{2n-k} : \Lambda^{2n-k}(\mathfrak{g}^*) \to \Lambda^k(\mathfrak{g})$$

defined by

$$G_\eta^k(u)(w) = \Phi_\eta^k(u, w) = \eta(u \wedge w), \quad G_\zeta^{2n-k}(\alpha)(\beta) = \Phi_\zeta^{2n-k}(\alpha, \beta) = (\alpha \wedge \beta)(\zeta)$$

respectively, are inverses of each other.

3. For $\alpha \in \Lambda^k(\mathfrak{g}^*)$, $\beta \in \Lambda^{2n-k}(\mathfrak{g}^*)$,

$$T_{G_\zeta^k(\alpha)}(\beta) = T_\zeta(\beta \wedge \alpha). \tag{5}$$

4. For $\alpha \in \Lambda^{2n-k}(\mathfrak{g}^*)$ and $\beta \in \Lambda^k(\mathfrak{g}^*)$,

$$\Phi_\eta^k(G_\zeta^{2n-k}(\alpha), G_\zeta^k(\beta)) = \Phi_\zeta^k(\beta, \alpha). \tag{6}$$

1.2. **Almost complex structures and almost Hermitian structures.** Let J be an almost complex structure on $\mathfrak{g}$. Then J induces a natural orientation on $\mathfrak{g}$, which we assume that it is the same as the fixed one.

Recall that J acts as an involution on $\Lambda^2(\mathfrak{g}^*)$, $\alpha(\cdot, \cdot) \to \alpha(J\cdot, J\cdot)$. And in Definition 0.3 we introduce $\mathcal{Z}_J^\pm = \mathcal{Z}^2 \cap \Lambda_J^\pm(\mathfrak{g}^*)$, and $H_J^\pm(\mathfrak{g})$ as the quotient

$$\frac{\mathcal{Z}^\pm}{\mathcal{B} \cap \Lambda_J^\pm(\mathfrak{g}^*)}. \tag{7}$$

Dually, J also acts on $\Lambda^2(\mathfrak{g})$ as an involution: $J(u \wedge v) = Ju \wedge Jv$. Accordingly, we also have the $J-$decomposition: $\Lambda^2(\mathfrak{g}) = \Lambda_J^+(\mathfrak{g}) \oplus \Lambda_J^-(\mathfrak{g})$, which gives rise to the subgroups

$H^J_\pm(\mathfrak{g})$ as the quotient

$$\frac{Z^\pm}{B\cap\Lambda^\pm_J(\mathfrak{g})}. \tag{8}$$

Let $\pi_\pm:\Lambda^2(\mathfrak{g})\to\Lambda^\pm_J(\mathfrak{g})$ be the projection, and

$$\Psi_\pm:\Lambda^\pm_J(\mathfrak{g})\times\Lambda^\pm_J(\mathfrak{g}^*)\to\mathbb{R}$$

be the restriction of Ψ^2 to $\Lambda^\pm_J(\mathfrak{g})\times\Lambda^\pm_J(\mathfrak{g}^*)$.

Lemma 1.6. *We have*

i). $(\Lambda^\pm_J(\mathfrak{g}),\Lambda^\mp_J(\mathfrak{g}^*))$ is a Ψ^2-complementary pair.

ii). $\Psi_\pm$ is non-degenerate.

iii). $\bar\Psi^2(H^\pm_J(\mathfrak{g}),H^J_\mp(\mathfrak{g}))=0$.

iv). $(\mathcal{Z}^\pm,\pi_\pm B^2)$ and $(\mathcal{B}\cap\Lambda^\pm_J(\mathfrak{g}^*),\pi_\pm Z^2)$ are $\Psi_\pm-$complementary pairs.

v). There is an induced isomorphism

$$\sigma:H^\pm_J(\mathfrak{g})=\frac{\mathcal{Z}^\pm_J}{\mathcal{B}\cap\Lambda^\pm_J(\mathfrak{g}^*)}\longrightarrow\hom(\frac{\pi_\pm Z^2}{\pi_\pm B^2},\mathbb{R}).$$

Proof. Since the $J-$actions on $\Lambda^2(\mathfrak{g})$ and $\Lambda^2(\mathfrak{g}^*)$ are involutions and satisfy (3) with respect to the pairing Ψ^2, (i) and (ii) follow from the second and the third bullets of Lemma 1.2.

iii) follows from i) and the description of $H^\pm_J(\mathfrak{g})$ in (7).

iv) follows easily from $(\alpha,\pi_\pm u)=(\alpha,u)$ for $\alpha\in\Lambda^\pm_J(\mathfrak{g}^*)$ and $u\in\Lambda^2(\mathfrak{g})$, and Lemma 1.3.

v) follows from iv) and the 4th bullet of Lemma 1.1. □

We note that, under $\bar\Psi^2$, the pairings between $H^\pm_J(\mathfrak{g})$ and $H^J_\pm(\mathfrak{g})$ may not be non-degenerate. It will be the case when $\mathfrak{g}$ is 4-dimensional and unimodular (Remark 3.5).

Let now h be an inner product on an oriented $2n$-dimensional Lie algebra $\mathfrak{g}$. Let $*_h:\Lambda^k(\mathfrak{g}^*)\to\Lambda^{2n-k}(\mathfrak{g}^*)$ be the Hodge-operator associated with h, namely, given $\beta\in\Lambda^k(\mathfrak{g}^*)$, for every $\alpha\in\Lambda^k(\mathfrak{g}^*)$ define $*_h\beta$ by the following relation

$$\alpha\wedge *_h\beta=\langle\alpha,\beta\rangle\mathrm{Vol}_h\,, \tag{9}$$

where $\langle,\rangle$ is the inner product on forms induced by h and Vol_h is the volume form of h with respect to the given orientation. Let $\eta=\mathrm{Vol}_h\in\Lambda^{2n}(\mathfrak{g}^*)$ and choose $\zeta\in\Lambda^{2n}(\mathfrak{g})$ with $\Psi^{2n}(\zeta,\eta)=1$. Then

$$\Phi^k_\zeta(\alpha,*_h\beta)=\langle\alpha,\beta\rangle. \tag{10}$$

For every $\alpha\in\Lambda^p(\mathfrak{g}^*)$ set, as usual,

$$\delta\alpha=(-1)^{n(p+1)+1}*_h\circ d\circ *_h\,.$$

Then, set $\Delta=d\delta+\delta d$ and denote by

$$\mathcal{H}^p(\mathfrak{g})=\{\alpha\in\Lambda^p(\mathfrak{g}^*)\mid\Delta\,\alpha=0\}\,,$$

the space of h-harmonic p-forms.

Let J be an almost complex structure on $\mathfrak{g}$. The inner product h is called *J-Hermitian* if J is an orthogonal transformation with respect to h. In this case, the pairing φ defined

by $\varphi(\cdot,\cdot) = h(J\cdot,\cdot)$ lies in $\Lambda^+_J(\mathfrak{g}^*)$, and it is called the fundamental form of (J,h). The pair (J,h), or equivalently, the triple (J,h,φ) is called an *almost Hermitian structure*.

With this understood, J is *almost Kähler* if there is a J-Hermitian h such that the associated fundamental form is closed.

1.3. **Complex structures.** Let $\mathfrak{g}$ be a $2n$-dimensional Lie algebra. An almost complex structure J on $\mathfrak{g}$ is said to be *integrable* if $N_J = 0$, where

$$N_J(u,v) = [u,v] - J[Ju,v] - J[u,Jv] - [Ju,Jv],$$

for all u, $v \in \mathfrak{g}$. Denoting by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathfrak{g}$, then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$, where $\mathfrak{g}^{1,0}$, $\mathfrak{g}^{0,1}$, respectively, are the $+\mathbf{i}$, $-\mathbf{i}$ eigenspaces of J acting on $\mathfrak{g}_{\mathbb{C}}$. Alternatively,

$$\mathfrak{g}^{1,0} = \{u - \mathbf{i}Ju \mid u \in \mathfrak{g}\}, \quad \mathfrak{g}^{0,1} = \{u + \mathbf{i}Ju \mid u \in \mathfrak{g}\}.$$

Similarly, $\mathfrak{g}^*_{\mathbb{C}}$ decomposes as $\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{0,1}$, where

$$\mathfrak{g}_{1,0} = \{\alpha \in \mathfrak{g}^*_{\mathbb{C}} \mid \alpha(z) = 0\,,\ \forall z \in \mathfrak{g}^{0,1}\}, \quad \mathfrak{g}_{0,1} = \{\alpha \in \mathfrak{g}^*_{\mathbb{C}} \mid \alpha(z) = 0\,,\ \forall z \in \mathfrak{g}^{1,0}\}$$

Accordingly, the space of complex r-forms $\Lambda^r(\mathfrak{g}^*_{\mathbb{C}})$ on $\mathfrak{g}^{\mathbb{C}}$ decomposes as

$$\Lambda^r(\mathfrak{g}^*_{\mathbb{C}}) = \bigoplus_{p+q=r} \Lambda^{p,q}_J(\mathfrak{g}^*_{\mathbb{C}}),$$

where $\Lambda^{p,q}_J(\mathfrak{g}^*_{\mathbb{C}}) = \Lambda^p(\mathfrak{g}_{1,0}) \wedge \Lambda^q(\mathfrak{g}_{0,1})$.
A straightforward computation shows that $N_J = 0$ if and only if $[\mathfrak{g}^{0,1}, \mathfrak{g}^{0,1}] \subset \mathfrak{g}^{0,1}$.
It is immediate to prove the following

Proposition 1.7. *J is integrable if there exists a non-zero closed $(n,0)$-form on $\mathfrak{g}$.*

It can be also proved that the integrability condition is equivalent to

$$d\mathfrak{g}_{1,0} \subset \mathfrak{g}_{1,1} \oplus \mathfrak{g}_{2,0},$$

where the differential d is extended by $\mathbb{C}$-linearity. Let J be an almost complex structure on $\mathfrak{g}$. Set $\overline{\partial}|_{\Lambda^{p,q}_J(\mathfrak{g}^*_{\mathbb{C}})} = \pi^{(p,q+1)} \circ d$. Then

$$\overline{\partial} : \Lambda^{p,q}_J(\mathfrak{g}^*_{\mathbb{C}}) \to \Lambda^{p,q+1}_J(\mathfrak{g}^*_{\mathbb{C}}),$$

and $\overline{\partial}$ can be extended by $\mathbb{C}$-linearity to $\Lambda^r(\mathfrak{g}^*_{\mathbb{C}})$. Then, J is integrable if and only if $\overline{\partial}^2 = 0$. This allows to define a Dolbeault complex for $\Lambda^{p,q}_J(\mathfrak{g}^*_{\mathbb{C}})$. We denote by $H^{*,*}_{\overline{\partial}}(\mathfrak{g})$ the cohomology of this complex. We refer to it as the *Dolbeault cohomology* of $\mathfrak{g}$.

1.4. **Tamed and almost Kähler structures in terms of cone of positive vectors.** We characterize tamed and almost Kähler almost complex structures on a Lie algebra in terms of the convex cone of positive vectors (compare with [29, Theorem 3.2], [17, Theorem 14] and [22]).

Clearly, any simple $2-$vector of the form $v \wedge Jv$ is invariant:

$$J(v \wedge Jv) = Jv \wedge JJv = -Jv \wedge v = v \wedge Jv.$$

Definition 1.8. *A simple $2-$vector is called* positive *if it is of the form $v \wedge Jv$. A vector in $\Lambda^+_J(\mathfrak{g})$ is said to be* positive *if it can be written as a convex combination of $v \wedge Jv$.*

Definition 1.9. *We denote by PC_J the convex cone of positive 2-vectors, and we set*

$$HPC_J = \{[u] \mid u \in PC_J \cap Z\},$$

which is a convex cone of $H^J_+(\mathfrak{g})$.

We will need the following finite dimensional Hahn-Banach lemma, which will be applied to the bilinear pairings Ψ^2 and Ψ^+ introduced in 1.1.

Lemma 1.10. *Let V be a finite dimensional vector space and W a linear subspace. If K is a bounded closed convex subset which is disjoint from W. Then there is a linear functional L vanishing on W and strictly positive on K.*

Let h be a J-Hermitian metric on $\mathfrak{g}$ and let φ be the fundamental 2-form of h. Set

$$K = \{w \in PC_J \mid T_w(\varphi) = 1\}.$$

Then $K \subset PC_J$ is bounded, closed and convex.

Proposition 1.11. *Let J be an almost complex structure on $\mathfrak{g}$.*

- *J is tamed if and only if $B^2 \cap PC_J = \{0\}$. There is a bijection between $J-$tamed forms and functionals on $\Lambda^2(\mathfrak{g})$ vanishing on B^2 and positive on $PC_J\backslash\{0\}$.*
- *J is almost Kähler if and only if $\pi_+B^2 \cap PC_J = \{0\}$. There is a bijection between $J-$compatible forms and functionals on $\Lambda^+_J(\mathfrak{g})$ vanishing on π_+B^2 and positive on $PC_J\backslash\{0\}$.*

Proof. First of all we prove the characterization of tamed J.

($\Longrightarrow$). Suppose J is tamed by a symplectic form ω. Then $\Psi^2(B^2, \omega) = 0$ by Lemma 1.3, and $\Psi^2(PC_J\backslash\{0\}, \omega) > 0$ by Definitions 0.1 and 1.8. Thus $B^2 \cap PC_J = \{0\}$.

($\Longleftarrow$). Conversely, suppose $B^2 \cap PC_J = \{0\}$. Then $B^2 \cap K = \emptyset$. By Lemma 1.10, there is a linear functional L on $\Lambda^2(\mathfrak{g})$, vanishing on B^2 and strictly positive on K. By Lemma 1.3, L determines a closed 2-form, which is positive on $PC_J\backslash\{0\}$. Such a form is a tamed symplectic form by Definition 0.1.

The argument for characterizing an almost Kähler J is similar. The only difference is that, instead of applying Lemma 1.3 for the pairing Ψ^2, we apply Lemma 1.6 (iv) for the pairing Ψ_+.

($\Longrightarrow$). Suppose J is compatible with a symplectic form ω. Then $\omega \in \mathcal{Z}^+$, and we have $\Psi_+(\pi_+B^2, \omega) = 0$ by Lemma 1.6 (iv). Moreover, $\Psi_+(PC_J\backslash\{0\}, \omega) > 0$ by Definitions 0.1 and 1.8, therefore $\pi_+B^2 \cap PC_J = \{0\}$.

($\Longleftarrow$). Conversely, suppose $\pi_+B^2 \cap PC_J = \{0\}$. Then $\pi_+B^2 \cap K = \emptyset$, and by Lemma 1.10, there is a linear functional L on $\Lambda^+_J(\mathfrak{g})$, vanishing on π_+B^2 and strictly positive on K. By Lemma 1.6 (iv), L determines a 2-form in $\mathcal{Z}^+$, which is positive on $PC_J\backslash\{0\}$. Such a form is a compatible symplectic form by Definition 0.1.

Finally, it is clear that both the bijection statements follow from the arguments above. □

Remark 1.12. *It has to be noted that complex structures which are tamed by a symplectic form are related to the so called* strong Kähler metrics with torsion *(shortly* SKT metrics*), also called in the literature as* pluriclosed *metrics. More precisely, a Hermitian metric on a complex Lie algebra is said to be SKT, if its fundamental form ω satisfies the condition*

$\partial\overline{\partial}\omega = 0$ *(the same definition can be given for an arbitrary complex manifold). The strong Kähler metrics with torsion have also applications in type II string theory and in 2-dimensional supersymmetric* σ*-models* [15, 28] *and are related to generalized Kähler structures (see* [16]*).*
It can be easily proved that if J *is a complex structure on a Lie algebra* $\mathfrak{g}$ *which is tamed by a symplectic form, then* $\mathfrak{g}$ *admits an SKT metric. For the sake of completeness we record the argument (see e.g.* [2, Proposition 3.4]*). Indeed, let* ω *be a symplectic form on* $\mathfrak{g}$ *taming* J*. Denote by* $\tilde{\omega}$ *the fundamental form of the Hermitian metric*

$$\tilde{g}_J(u,v) = \frac{1}{2}\left(\omega(u,Jv) + \omega(v,Ju)\right).$$

Then we easily obtain that

$$\partial\overline{\partial}\,\tilde{\omega} = 0.$$

Indeed, by definition, we have

$$\tilde{\omega} = \frac{1}{2}\left(\omega + J\omega\right).$$

Then, we can write

$$\begin{array}{rcl} \omega & = & \omega^{2,0} + \omega^{1,1} + \overline{\omega^{2,0}}\,, \\ \tilde{\omega} & = & \omega^{1,1}\,, \end{array}$$

where $\overline{\omega^{1,1}} = \omega^{1,1}$*. Consequently*

$$d\omega = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{rl} \partial\omega^{2,0} & = 0\,, \\ \partial\omega^{1,1} + \overline{\partial}\omega^{2,0} & = 0\,, \end{array} \right.$$

since $d = \partial + \overline{\partial}$*. Therefore,*

$$\partial\overline{\partial}\tilde{\omega} = \partial\overline{\partial}\omega^{1,1} = -\overline{\partial}\partial\omega^{1,1} = \overline{\partial}^2\omega^{2,0} = 0\,.$$

2. Tamed versus almost Kähler in dimension 4

In this section we prove Theorem 0.2. We begin with reviewing the special feature of $\Lambda^2(\mathfrak{g})$ in dimension 4.

2.1. **Geometry of** $\Lambda^2(\mathfrak{g})$**.** Suppose $\mathfrak{g}$ is a 4−dimensional oriented Lie algebra. In this case $\Lambda^2(\mathfrak{g})$ and $\Lambda^2(\mathfrak{g}^*)$ have dimension 6. Fix a basis $\{f_1,\dots,f_4\}$ of $\mathfrak{g}$ and denote by $\{f^1,\dots,f^4\}$ the dual basis of $\{f_1,\dots,f_4\}$. Then $\{f_i\wedge f_j\}$ and $\{f^i\wedge f^j\}$ are bases of $\Lambda^2(\mathfrak{g})$ and $\Lambda^2(\mathfrak{g}^*)$.

Let ζ be a nonzero 4−vector on $\mathfrak{g}$ inducing the given orientation, and η a volume form with $\Psi^4(\zeta,\eta) = 1$. For ease of notations, denote the pairings $\Psi^2, \Phi^2_\zeta, \Phi^2_\eta$ by $\Psi, \Phi_\zeta, \Phi_\eta$ respectively. Explicitly,

$$\Psi : \Lambda^2(\mathfrak{g}) \times \Lambda^2(\mathfrak{g}^*) \to \mathbb{R}, \quad \Phi_\zeta : \Lambda^2(\mathfrak{g}^*) \times \Lambda^2(\mathfrak{g}^*) \to \mathbb{R}, \quad \Phi_\eta : \Lambda^2(\mathfrak{g}) \times \Lambda^2(\mathfrak{g}) \to \mathbb{R}.$$

Similarly denote $G^2_\eta : \Lambda^2(\mathfrak{g}) \to \Lambda^2(\mathfrak{g}^*)$ by G_η, and $G^2_\zeta : \Lambda^2(\mathfrak{g}^*) \to \Lambda^2(\mathfrak{g})$ by G_ζ.

Let S denote the set of simple 2−vectors on $\mathfrak{g}$. The following easy observation will be useful.

Lemma 2.1. *Suppose $\mathfrak{g}$ has dimension 4. Then $u \in \Lambda^2(\mathfrak{g})$ is simple, i.e. $u \in S$ if and only if $\Phi_\eta(u,u) = 0$.*

There is a similar characterization of simple 2-forms. Thus $(\Lambda^2(\mathfrak{g}), \Phi_\eta)$ and $(\Lambda^2(\mathfrak{g}^*), \Phi_\zeta)$ have signature $(3,3)$.

Let h be an inner product on $\mathfrak{g}$. In dimension 4, the Hodge operator $*_h$ on $\Lambda^2(\mathfrak{g}^*)$ is another involution on $\Lambda^2(\mathfrak{g}^*)$ and induces the well known decomposition

$$\Lambda^2(\mathfrak{g}^*) = \Lambda_h^+(\mathfrak{g}^*) \oplus \Lambda_h^-(\mathfrak{g}^*), \tag{11}$$

where $\Lambda_h^+(\mathfrak{g}^*)$, $\Lambda_h^+(\mathfrak{g}^*)$ are the vector spaces of self-dual, anti-self-dual 2-forms on $\mathfrak{g}$ respectively. Since $\langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle$ it follows from the formula (10) that

$$\Phi_\zeta(\alpha,\beta) = \Phi_\zeta(*_h\alpha, *_h\beta). \tag{12}$$

Thus it follows from Lemma 1.2 that $\Lambda_h^+(\mathfrak{g}^*)$ and $\Lambda_h^-(\mathfrak{g}^*)$ are a $\Phi_\zeta-$complementary pair, and Φ_ζ is non-degenerate on $\Lambda_h^\pm(\mathfrak{g}^*)$. In fact, Φ_ζ is $\pm-$definite on $\Lambda_h^\pm(\mathfrak{g}^*)$: for $\alpha \in \Lambda_h^\pm(\mathfrak{g}^*)$,

$$\Phi_\zeta(\alpha,\alpha) = \Phi_\zeta(\alpha, \pm *_h \alpha) = \langle\alpha, \pm *_h *_h\alpha\rangle = \pm|\alpha|^2. \tag{13}$$

Since $(\Lambda^2(\mathfrak{g}^*), \Phi_\zeta)$ has signature $(3,3)$, $\Lambda_h^\pm(\mathfrak{g}^*)$ has dimension 3.

Let J be an almost complex structure on $\mathfrak{g}$. Then it follows from Lemma 1.2 that $\Lambda_J^+(\mathfrak{g}^*)$ and $\Lambda_J^-(\mathfrak{g}^*)$ are a $\Phi_\zeta-$complementary pair, and Φ_ζ is non-degenerate on $\Lambda_J^\pm(\mathfrak{g}^*)$. The same is true for $\Lambda_J^\pm(\mathfrak{g})$ with respect to Φ_η.

Suppose further h is J-Hermitian and $\varphi \in \Lambda^2(\mathfrak{g}^*)$ is the fundamental form of (J,h). In this case, $\eta = \frac{1}{2}\varphi\wedge\varphi$. One can easily verify the following relations among the subspaces $\Lambda_J^\pm(\mathfrak{g}^*), \Lambda_h^\pm(\mathfrak{g}^*), \mathrm{Span}\{\varphi\}$ of $\Lambda^2(\mathfrak{g}^*)$:

$$\Lambda_J^+(\mathfrak{g}^*) = \mathbb{R}(\varphi) \oplus \Lambda_h^-(\mathfrak{g}^*), \quad \Lambda_h^+(\mathfrak{g}^*) = \mathbb{R}(\varphi) \oplus \Lambda_J^-(\mathfrak{g}^*), \tag{14}$$

and

$$\Lambda_J^+(\mathfrak{g}^*) \cap \Lambda_h^+(\mathfrak{g}^*) = \mathbb{R}(\varphi), \quad \Lambda_J^-(\mathfrak{g}^*) \cap \Lambda_h^-(\mathfrak{g}^*) = \{0\}. \tag{15}$$

The following lemma is straightforward.

Lemma 2.2. *Suppose $\mathfrak{g}$ has dimension 4 and J is an almost complex structure on $\mathfrak{g}$. Consider the pairings Φ_η, Φ_ζ and the isomorphisms G_η, G_ζ. We have*

(i) $G_\zeta(\Lambda_J^\pm(\mathfrak{g}^*)) = \Lambda_J^\pm(\mathfrak{g})$, and $\Phi_\eta(G_\zeta(\alpha), G_\zeta(\beta)) = \Phi_\zeta(\beta,\alpha)$.

(ii) $\Lambda_J^+(\mathfrak{g})$ and $\Lambda_J^+(\mathfrak{g}^*)$ have signature $(1,3)$.

(iii) $\Lambda_J^-(\mathfrak{g})$ and $\Lambda_J^-(\mathfrak{g}^*)$ are positive definite.

(iv) If $v \in PC_J$, then $\Phi_\eta(v,v) \geq 0$.

Proof. (i) The first formula follows from the above mentioned fact that $(\Lambda_J^+(\mathfrak{g}^*), \Lambda_J^-(\mathfrak{g}^*))$ is a $\Phi_\zeta-$complementary pair, and the first part of Lemma 1.6. The second is a special case of the fourth part of Lemma 1.5.
(ii) and (iii) follow from (14) and (i).
(iv) follows from $\Phi_\eta(f_i \wedge Jf_i, f_j \wedge Jf_j) \geq 0$ for any i, j, which is straightforward. $\square$

2.2. **Tamed versus almost Kähler.** We will apply Proposition 1.11 to prove Theorem 0.2. The following observations on the set S of simple vectors are crucial for this purpose.

Lemma 2.3. *For an almost complex structure J on $\mathfrak{g}$ of arbitrary dimension, we have*

$$S \cap \Lambda_J^+(\mathfrak{g}) \subset PC_J \sqcup -PC_J. \tag{16}$$

Proof. Observe that if a simple 2-vector is in $\Lambda_J^+(\mathfrak{g})$ then it is of the form $\pm(v \wedge Jv)$, and in fact, $\Lambda_J^+(\mathfrak{g})$ is generated by simple vectors of the form $v \wedge Jv$. □

The second one is only valid in dimension 4.

Lemma 2.4. *Suppose $\mathfrak{g}$ has dimension 4. If $v \neq 0 \in S$ and $v \notin \Lambda_J^+(\mathfrak{g})$, then $\pi_+ v \notin PC_J$.*

Proof. Let $v \in S$ and $v = \pi_+ v + \pi_- v$. By assumption, $\pi_- v \neq 0$, and hence $\Phi_\eta(\pi_- v, \pi_- v) > 0$ by Lemma 2.2 (iii). Since

$$0 = \Phi_\eta(v, v) = \Phi_\eta(\pi_+ v, \pi_+ v) + \Phi_\eta(\pi_- v, \pi_- v),$$

$\Phi_\eta(\pi_+ v, \pi_+ v) < 0$. The conclusion follows from Lemma 2.2 (iv). □

We are ready to show the following theorem, from which Theorem 0.2 follows.

Theorem 2.5. *Let $\mathfrak{g}$ be a $4-$dimensional Lie algebra with $B \wedge B = 0$. Let J be an almost complex structure on $\mathfrak{g}$. The following are equivalent.*

0) *J is tamed.*
1) $B \cap PC_J = \{0\}$.
2) $B \cap \Lambda_J^+(\mathfrak{g}) = \{0\}$.
3) $\pi_+ B \cap PC_J = \{0\}$.
4) *J is almost Kähler.*

Proof. 0) = 1) is the first bullet of Proposition 1.11.

1) = 2) Since $PC_J \subset \Lambda_J^+(\mathfrak{g})$ by definition, we need to show only ($\Longrightarrow$), namely, if there is a nonzero vector in $B \cap \Lambda_J^+(\mathfrak{g})$, then there is a nonzero vector in $B \cap PC_J$.
Let $0 \neq v \in B \cap \Lambda_J^+(\mathfrak{g})$. Then by our assumption $v \wedge v = 0$. Now it follows from Lemma 2.1 that $v \in S$. Therefore $v \in S \cap \Lambda_J^+(\mathfrak{g})$. Hence, by Lemma 2.3, v or $-v \in PC_J$. Thus $B \cap PC_J$ contains either v or $-v$.

2) $\Longrightarrow$ 3) Suppose $0 \neq v \in B$, then $v \notin \Lambda_J^+(\mathfrak{g})$ by assumption. Since $v \in S$, by Lemma 2.4, $\pi_+ v \notin PC_J$. Therefore $\pi_+ B \cap PC_J = \{0\}$.

3) = 4) is the second bullet of Proposition 1.11.

4) $\Longrightarrow$ 0) An almost Kähler J is tamed by definition. □

It is easy to see that $B \wedge B = 0$ for any unimodular $4-$dimensional Lie algebra (see part 3 of Lemma 3.2).
In [12] it is proved that a 4-dimensional Lie algebra $\mathfrak{g}$ endowed with a complex structure J admits a taming symplectic structure if and only if $(\mathfrak{g}, J)$ has a Kähler metric. It is interesting to study whether Theorem 2.5 offers an alternative proof of this result.

2.3. **Positive $2-$planes and almost complex structures.** We have shown that on a $4-$dimensional unimodular Lie algebra, all tamed J are almost Kähler. In the next section we will determine which J is tamed. For this purpose we first review in this subsection Donaldson's description of $4-$dimensional almost complex structures in terms of positive $2-$planes in $\Lambda^2(\mathfrak{g}^*)$ ([8]). Then we calculate explicitly the positive $2-$planes $J \to \Lambda^-_J(\mathfrak{g}^*)$ given a basis of $\mathfrak{g}$.

Notice that both $\mathrm{SL}(4,\mathbb{R})$ and $\mathrm{SO}(3,3)$ have dimension 15. In fact, there is a natural representation $\pi : \mathrm{SL}(4,\mathbb{R}) \to \mathrm{Aut}(\Lambda^2(\mathfrak{g}), \Phi_\eta)$ of $\mathrm{SL}(4,\mathbb{R})$ on $\Lambda^2(\mathfrak{g})$: given any $A \in \mathrm{SL}\,(4,\mathbb{R})$, set $\pi(A)(\sum p^{ik} f_i \wedge f_k) = \sum p^{ik} Af_i \wedge Af_k$.

It is immediate to check that $\pi : \mathrm{SL}\,(4,\mathbb{R}) \to \mathrm{Aut}\,(\Lambda^2(\mathfrak{g}))$ is a group homomorphism, and $\pi(A)$ preserves Φ_η since $\det(A) = 1$. Since $\mathrm{SL}\,(4,\mathbb{R})$ is connected, $\pi(\mathrm{SL}\,(4,\mathbb{R})) \subset \mathrm{Aut}^0(\Lambda^2(\mathfrak{g}), \Phi_\eta) \simeq \mathrm{SO}^0(3,3)$, where the superscript 0 means the connected component of the identity. Furthermore, we have that $\mathrm{Ker}\,(\pi) = \pm I$. Indeed, let $A \in \mathrm{Ker}\,(\pi)$; then $\pi(A)(f_i \wedge f_j) = (Af_i \wedge Af_j) = f_i \wedge f_j\,, i,j = 1,\dots,4$. Hence, by part 4 of Lemma 1.5, for any fixed i, the vector $Af_i \in \mathrm{Span}\{f_i, f_j\}\,, j = 1,\dots,4$. Consequently, $A = \lambda I$. Since $A \in \mathrm{SL}(4,\mathbb{R})$, it follows that $\lambda = \pm 1$, i.e., $A = \pm I$.

Dualizing $\pi : \mathrm{SL}\,(4,\mathbb{R}) \to \mathrm{SO}^0(\Lambda^2(\mathfrak{g}), \Phi_\eta)$ we obtain the double covering $\pi^* : \mathrm{SL}\,(4,\mathbb{R}) \to \mathrm{SO}^0(\Lambda^2(\mathfrak{g}^*), \Phi_\zeta)$. One important consequence is that $\mathrm{SL}(4,\mathbb{R})$ acts transitively on the set of $\Phi_\zeta-$positive definite $k-$planes for $k = 1,2,3$. As observed by Donaldson in [8], such $k-$planes correspond to various geometric structures. In particular,

Proposition 2.6. ([8]) *For a $4-$dimensional Lie algebra $\mathfrak{g}$, the map $J \to \Lambda^-_J(\mathfrak{g}^*)$ is a one-to-one correspondence between the space of almost complex structures and the space of oriented positive definite two plane in $\Lambda^2(\mathfrak{g}^*)$.*

This point of view is useful to detect integrable J.

Lemma 2.7. *Suppose* $\dim \mathfrak{g} = 4$. *Then J is integrable if $\Lambda^-_J(\mathfrak{g}^*) \subset \mathcal{Z}$.*

Proof. Suppose $\alpha \in \Lambda^-_J$. Let us consider the additional action of s_J of J on $\Lambda^2(\mathfrak{g}^*)$, defined as

$$s_J\alpha(u,v) = \alpha(Ju,v).$$

This action preserves Λ^-_J. Then, by the assumption, both α and $s_J\alpha$ are closed. Therefore $\alpha + \mathbf{i}(s_J\alpha)$ is a closed $(2,0)-$form and the statement follows from Proposition 1.7. □

Fix a basis $\{f_i\}$ of $\mathfrak{g}$. We present an explicit covering of the 8-dimensional space of almost complex structures, and an explicit calculation of $\Lambda^-_J(\mathfrak{g}^*)$ with respect to this covering.

Suppose

$$Jf_k = \sum_{h=1}^{4} a_{hk} f_h\,, \quad a_{hk} \in \mathbb{R}.$$

Assume that $\{f_1, Jf_1, f_3, Jf_3\}$ is still a basis of $\mathfrak{g}$. Then

$$\Lambda^-_J(\mathfrak{g}^*) = \mathrm{Span}\left\{ f^{24} - \sum_{h<k} p_{hk} f^{hk}\,,\ \sum_h \left(a_{2h} f^{4h} - a_{4h} f^{2h} \right) \right\}. \tag{17}$$

where

$$p_{hk} = \det \begin{pmatrix} a_{2h} & a_{2k} \\ a_{4h} & a_{4k} \end{pmatrix} \qquad h,k = 1,\dots,4\,, h<k,$$

are the Plücker coordinates of the vectors (a_{2i}) and (a_{4j}). The previous computation is general. Indeed, the condition that $\{f_1, Jf_1, f_3, Jf_3\}$ is a basis is equivalent to say that the 2-plane $L_{13} = \mathrm{Span}\{f_1, f_3\}$ is not J-invariant. Set, for any pair of indices (i,j), with $i<j$, $L_{ij} = \mathrm{Span}\{f_i, f_j\}$ and

$$\mathcal{U}_{ij} = \{J \mid L_{ij} \text{ is not } J-\text{invariant}\}.$$

Since L_{ij} cannot be $J-$invariant for all pairs, $\{\mathcal{U}_{ij}\}$ is a covering of the space of the almost complex structures on $\mathfrak{g}$. It can be proved that

$$\mathcal{U}_{ij} \simeq \{(a_i, a_j) \in \mathbb{R}^4 \times \mathbb{R}^4 \,|\, p_{\bar{i}\bar{j}} = \det \begin{pmatrix} a_{\bar{i}i} & a_{\bar{i}j} \\ a_{\bar{j}i} & a_{\bar{j}j} \end{pmatrix} \neq 0 \text{ where}, \bar{i},\, \bar{j} \in \{1,\dots,4\}\setminus\{i,j\},\, \bar{i} < \bar{j}\}.$$

3. Unimodular 4-dimensional Lie algebras

In this section $\mathfrak{g}$ is a unimodular Lie algebra.

3.1. **Unimodularity.** We start by recalling the following

Definition 3.1. *A $2n$-dimensional Lie algebra $\mathfrak{g}$ is said to be* unimodular *if* $\mathrm{tr}(\mathrm{ad}_\xi) = 0$ *for every $\xi \in \mathfrak{g}$.*

It turns out that $\mathfrak{g}$ is unimodular if and only if

$$d(\Lambda^{2n-1}(\mathfrak{g}^*)) = 0 \tag{18}$$

(see [19, p. 81 (6.3)]). Equivalently, $\mathfrak{g}$ is unimodular if and only if $b_{2n}(\mathfrak{g}) = 1$.

Lemma 3.2. *Let $\mathfrak{g}$ be a $2n$-dimensional unimodular Lie algebra. Then*

1. $G^k_\eta(Z_k) = \mathcal{Z}^{2n-k}$.
2. $G^k_\eta(B_k) = \mathcal{B}^{2n-k}$.
3. $(\mathcal{Z}^k, \mathcal{B}^{2n-k})$ *and* (Z^k, B^{2n-k}) *are* Φ^k_ζ*-complementary pairs.*
4. *There are induced non-degenerate pairings*

$$\bar{\Phi}^k_\zeta : H^k(\mathfrak{g}) \times H^{2n-k}(\mathfrak{g}) \to \mathbb{R}, \quad \bar{\Phi}^k_\eta : H_k(\mathfrak{g}) \times H_{2n-k}(\mathfrak{g}) \to \mathbb{R}.$$

5. *Furthermore, let h be an inner product on $\mathfrak{g}$ with the volume form* Vol_h. *Then, for every $\alpha \in \Lambda^{p-1}(\mathfrak{g}^*)$, $\beta \in \Lambda^p(\mathfrak{g}^*)$ we have*

$$\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle,$$

and the following Hodge decomposition holds

$$\Lambda^p(\mathfrak{g}^*) = \mathcal{H}^p(\mathfrak{g}) \oplus d(\Lambda^{p-1}(\mathfrak{g}^*)) \oplus \delta(\Lambda^{p+1}(\mathfrak{g}^*))\,. \tag{19}$$

Proof. 1) Let $u \in Z_k$ be a k-cycle and let $\alpha = G^k_\eta(u) \in \Lambda^{2n-k}(\mathfrak{g}^*)$. Then, for every $\gamma \in \Lambda^{k-1}(\mathfrak{g}^*)$, $d(\gamma\wedge\alpha) = 0$ since $\mathfrak{g}$ is unimodular. By applying the third part of lemma 1.5, we have

$$T_\zeta(\gamma \wedge d\alpha) = (-1)^k T_\zeta\,(d\gamma\wedge\alpha) = (-1)^k T_{G^{2n-k}_\zeta(\alpha)}(d\gamma) = (-1)^k T_u(d\gamma) = -dT_u(\gamma) = 0,$$

and, consequently, $d\alpha = 0$.

2) Let $u \in B^k$ be a k-boundary, i.e., $T_u = dT_w$, for a $(k+1)-$vector w.
Set $\beta = G_\eta^{k+1}(w) \in \Lambda^{2n-k-1}(\mathfrak{g}^*)$. Then, by applying the third part of lemma 1.5 twice, for every $\gamma \in \Lambda^k(\mathfrak{g}^*)$, we have

$$\begin{aligned} T_\zeta(\gamma \wedge \alpha) &= T_{G_\zeta^{2n-k}(\alpha)}(\gamma) = T_u(\gamma) = dT_w(\gamma) = (-1)^{k+1}T_w(d\gamma) = (-1)^{k+1}T_{G_\zeta^{2n-k-1}(\beta)}(d\gamma) \\ &= (-1)^{k+2}T_\zeta(d\gamma \wedge \beta) = -T_\zeta(\gamma \wedge d\beta); \end{aligned}$$

therefore, $\alpha = d(-\beta)$.

3) By part 4 of Lemma 1.5 and 1) and 2) it suffices to prove $\mathcal{Z}^k$ and $\mathcal{B}^{2n-k}$ are Φ_ζ^k-complementary pair. Further, by Lemma 1.1 it is enough to prove that

$$\mathcal{Z}^k = \left(\mathcal{B}^{2n-k}\right)^\perp,$$

where $\perp$ is taken with respect to Φ_ζ^k.

Let $\alpha \in \mathcal{Z}^k$ and $\beta \in \mathcal{B}^{2n-k}$, $\beta = d\gamma$; then, we have

$$\Phi_\zeta^k(\alpha, d\gamma) = (\alpha \wedge d\gamma)(\zeta) = (-1)^{k+1}(d\alpha \wedge \gamma)(\zeta) = 0,$$

i.e., $\Phi_\zeta^k(\mathcal{Z}^k, \mathcal{B}^{2n-k}) = 0$. Therefore, $\mathcal{Z}^k \subset \left(\mathcal{B}^{2n-k}\right)^\perp$.
Conversely, let $\alpha \in \left(\mathcal{B}^{2n-k}\right)^\perp$. Then, for any $\gamma \in \Lambda^{2n-k-1}(\mathfrak{g}^*)$, we have

$$0 = \Phi_\zeta^k(\alpha, d\gamma) = (\alpha \wedge d\gamma)(\zeta) = (-1)^{k+1}(d\alpha \wedge \gamma)(\zeta);$$

hence, $(d\alpha \wedge \gamma)(\zeta) = 0$, for every $\gamma \in \Lambda^{2n-k-1}(\mathfrak{g}^*)$.
Therefore, $d\alpha = 0$ and $\left(\mathcal{B}^{2n-k}\right)^\perp \subset \mathcal{Z}^k$.

4) It follows from 3) and the last bullet of Lemma 1.1.

5) Consider the linear map $T_\zeta : \Lambda^{2n}(\mathfrak{g}^*) \to \mathbb{R}$ with ζ satisfying $\mathrm{Vol}_h(\zeta) = 1$. Since $\alpha \wedge *_h\beta$ is a $(2n-1)$-form and $\mathfrak{g}$ is unimodular, $d(\alpha \wedge *_h\beta) = 0$. Thus we have

$$0 = T_\zeta\left(d(\alpha \wedge *_h\beta)\right) = \langle d\alpha, \beta\rangle - \langle \alpha, \delta\beta\rangle.$$

This implies that Δ is self-adjoint, and $\Delta\,\alpha = 0$ if and only if $d\alpha = 0 = \delta\alpha$. Since the complex $(\Lambda^*(\mathfrak{g}), d)$ has finite dimension and Δ is self-adjoint, it follows at once that

$$\Lambda^p(\mathfrak{g}^*) = \mathrm{Ker}\,\Delta \oplus \mathrm{Im}\,\Delta,$$

which implies (19). □

3.2. **Cohomological J-decompositions.** Let $\mathfrak{g}$ be a 4-dimensional unimodular Lie algebra endowed with an inner product h. Let $\mathcal{H}(\mathfrak{g})$ be the space of h-harmonic 2-forms on $\mathfrak{g}$. Then, by (19), we have the Hodge decomposition

$$\Lambda^2(\mathfrak{g}^*) = \mathcal{H}(\mathfrak{g}) \oplus d(\Lambda^1(\mathfrak{g}^*)) \oplus d^*(\Lambda^3(\mathfrak{g}^*)). \tag{20}$$

Let $\mathcal{H}^+(\mathfrak{g})$, $\mathcal{H}^-(\mathfrak{g})$ be the space of h-harmonic self-dual, anti-self-dual forms respectively on $\mathfrak{g}$. Since $*_h$ commutes with Δ we obtain that

$$\mathcal{H}(\mathfrak{g}) = \mathcal{H}^+(\mathfrak{g}) \oplus \mathcal{H}^-(\mathfrak{g})$$

Set

$$H_h^{\pm}(\mathfrak{g}) = \{a \in H^2(\mathfrak{g}) \mid \text{there exists}\, \alpha \in \mathcal{Z}_h^{\pm} \text{ such that}\, a = [\alpha]\}\,,$$

where $\mathcal{Z}_h^{\pm} = \mathcal{Z}^2 \cap \Lambda_h^{\pm}(\mathfrak{g})$. Then, by (20), the inclusion map $\mathcal{H}(\mathfrak{g}) \hookrightarrow \Lambda^2(\mathfrak{g}^*)$ gives rise to an isomorphism between $\mathcal{H}(\mathfrak{g})$ and $H^2(\mathfrak{g})$ such that $\mathcal{H}^+(\mathfrak{g}) \cong H_h^+(\mathfrak{g})$, $\mathcal{H}^-(\mathfrak{g}) \cong H_h^-(\mathfrak{g})$ and

$$H^2(\mathfrak{g}) = H_h^+(\mathfrak{g}) \oplus H_h^-(\mathfrak{g})\,.$$

Let us consider the cohomology groups $H_J^+(\mathfrak{g})$, $H_J^-(\mathfrak{g})$ associated with an almost complex structure J.

Theorem 3.3. *Let $\mathfrak{g}$ be a unimodular 4-dimensional Lie algebra $\mathfrak{g}$ endowed with an almost complex structure J. Then the following decomposition holds:*

$$H^2(\mathfrak{g}) = H_J^+(\mathfrak{g}) \oplus H_J^-(\mathfrak{g}). \tag{21}$$

Proof. First of all we show that $H_J^+(\mathfrak{g}) \cap H_J^-(\mathfrak{g}) = \{0\}$. Indeed, let $0 \neq \mathfrak{a} \in H_J^+(\mathfrak{g}) \cap H_J^-(\mathfrak{g})$. Then $\mathfrak{a} = [\alpha] = [\beta]$, where $\alpha \in \mathcal{Z}_J^+$, $\beta \in \mathcal{Z}_J^-$ respectively and $\beta = \alpha + d\gamma$. Let $h = \langle , \rangle$ be a J-Hermitian inner product on $\mathfrak{g}$, denote by φ the fundamental form of (J, h). Since the inner product is J-invariant, it follows immediately that $\langle \alpha, \beta \rangle = 0$. On the other hand, by (14), we have that $\Lambda_J^-(\mathfrak{g}^*) \subset \Lambda_h^+(\mathfrak{g}^*)$. Consequently, β is self-dual harmonic. Then,

$$0 < |\beta|^2 = \langle \beta, \beta \rangle = \langle \beta, \alpha + d\gamma \rangle = \langle \beta, d\gamma \rangle = 0\,.$$

Hence $\mathfrak{a} = 0$.

Now we have to show that

$$H^2(\mathfrak{g}) = H_J^+(\mathfrak{g}) + H_J^-(\mathfrak{g}). \tag{22}$$

Assume on the contrary that (22) is not true. Then we can find a cohomology class $0 \neq \mathfrak{a} \in H^2(\mathfrak{g})$ such that $\mathfrak{a}$ is $\bar{\Phi}_\zeta-$orthogonal to $H_J^+(\mathfrak{g}) + H_J^-(\mathfrak{g})$. We may assume that $\mathfrak{a} \in H_h^+(\mathfrak{g})$, since $H_h^-(\mathfrak{g}) \subset H_J^+(\mathfrak{g})$. Let α be the h-harmonic representative of $\mathfrak{a}$. Since $\mathfrak{a} \in H_h^+(\mathfrak{g})$, α is self-dual. Set $c = \langle \alpha, \varphi \rangle$. Then $c \neq 0$, otherwise $a \in H_J^-(\mathfrak{g})$.

We will use the following decomposition properties, which are the Lie algebraic counterpart of Lemma 2.4 of [9]: if $\alpha \in \Lambda_h^+(\mathfrak{g})$ and $\alpha = \alpha_{harm} + d\lambda + \delta\eta$ is its Hodge decomposition according to (20), then

$$(d\lambda)_h^+ = (\delta\eta)_h^+ \quad (d\lambda)_h^- = -(\delta\eta)_h^-\,, \tag{23}$$

$$\alpha - 2(d\lambda)_h^+ = \alpha_{harm}, \tag{24}$$

$$\alpha + 2(d\lambda)_h^- = \alpha_{harm} + 2d\lambda, \tag{25}$$

where, according to the decomposition (11), any $\beta \in \Lambda^2(\mathfrak{g}^*)$ can be written as $\beta = \beta_h^+ + \beta_h^-$ with $\beta_h^{\pm} \in \Lambda_h^{\pm}(\mathfrak{g}^*)$.

For the sake of completeness, we give the proof of (23), (24) (the proof of (25) being similar). Denote $*_h$ by $*$. Since $*\alpha = \alpha$, we get

$$*\alpha_{harm} + *d\lambda + *\delta\eta = \alpha_{harm} + d\lambda + \delta\eta.$$

By the uniqueness of Hodge decomposition (19), we obtain

$$*\, d\lambda = \delta\eta\,, \quad *\delta\eta = d\lambda. \tag{26}$$

Therefore, by using (26), we get

$$(d\lambda)_h^+ = \frac{1}{2}(d\lambda + *d\lambda) = \frac{1}{2}(*\delta\eta + \delta\eta) = (\delta\eta)_h^+$$

and

$$(d\lambda)_h^- = \frac{1}{2}(d\lambda - *d\lambda) = \frac{1}{2}(*\delta\eta - \delta\eta) = -(\delta\eta)_h^-$$

i.e., (23) is proved.

By using again (26), we have

$$\alpha - 2(d\lambda)_h^+ = \alpha - (d\lambda + *d\lambda) = \alpha_{harm}$$

and (24) is proved.

Now, applying (25) to the self-dual form $c\,\varphi$, we obtain:

$$c\,\varphi + 2((c\,\varphi)_{exact})_h^- = (c\,\varphi)_{harm} + 2(c\,\varphi)_{exact} \in \mathcal{Z}_J^+\,,$$

where $(\alpha)_{exact}$ denotes the exact part from of the form α from the Hodge decomposition. Setting $\mathfrak{b} = [(c\,\varphi)_{harm}] \in H_J^+(\mathfrak{g})$, we get

$$0 = \overline{\Phi}_\zeta(\mathfrak{a}, \mathfrak{b}) = \Phi_\zeta(\alpha, (c\,\varphi)_{harm} + 2(c\,\varphi)_{exact}) = \Phi_\zeta(\alpha, c\,\varphi + 2((c\,\varphi)_{exact})_h^-) = c^2.$$

This is absurd, since $c \neq 0$. □

Remark 3.4. *If either the condition being $4-$dimensional or unimodular is dropped, then* (22) *may not hold. For higher dimensional examples see* [13].

Remark 3.5. *Theorem 3.3 implies that the pairings between $H_J^\pm(\mathfrak{g})$ and $H^J_\pm(\mathfrak{g})$ are non-degenerate, and there is also a homological $J-$decomposition: $H_2(\mathfrak{g}) = H^J_+(\mathfrak{g}) \oplus H^J_-(\mathfrak{g})$. Indeed,*

- *$H^J_+(\mathfrak{g}) \cap H^J_-(\mathfrak{g}) = \{0\}$. Let $\overline{\Psi} : H_2(\mathfrak{g}) \times H^2(\mathfrak{g}) \to \mathbb{R}$ be the non-degenerate pairing as in Lemma 1.3. Let $[u] \in H^J_+(\mathfrak{g}) \cap H^J_-(\mathfrak{g})$. Then*

$$\overline{\Psi}([u], \cdot)_{|_{H_J^+(\mathfrak{g})}} = 0\,, \quad \overline{\Psi}([u], \cdot)_{|_{H_J^-(\mathfrak{g})}} = 0.$$

  Since $H^2(\mathfrak{g}) = H_J^+(\mathfrak{g}) + H_J^-(\mathfrak{g})$ and $\overline{\Psi}$ is non-degenerate, it follows that $[u] = 0$.
- *$H_2(\mathfrak{g}) = H^J_+(\mathfrak{g}) + H^J_-(\mathfrak{g})$. Let $\bar{\Phi}_\zeta : H^2(\mathfrak{g}) \times H^2(\mathfrak{g}) \to \mathbb{R}$ be the non-degenerate pairing as in Lemma 3.2 and let $\bar{\Psi} : \hom(H^2(\mathfrak{g}), \mathbb{R}) \to H_2(\mathfrak{g})$ be the isomorphism induced by the non-degenerate pair $\bar{\Psi}$ of Lemma 1.3. Then, $\bar{\Psi} \circ \bar{\Phi}_\zeta : H^2(\mathfrak{g}) \to H_2(\mathfrak{g})$ is an isomorphism. In fact this map is induced by $G_\zeta : \Lambda^2(\mathfrak{g}^*) \to \Lambda^2(\mathfrak{g})$. It then follows from the properties of G_ζ in the first part of Lemma 2.2 and parts 1 and 2 of Lemma 3.2 that $\bar{\Psi} \circ \bar{\Phi}_\zeta$ induces inclusions $H_J^\pm(\mathfrak{g}) \hookrightarrow H^J_\pm(\mathfrak{g})$. Since $H^2(\mathfrak{g})$ and $H_2(\mathfrak{g})$ have the same dimension, we have $H_2(\mathfrak{g}) = H^J_+(\mathfrak{g}) \oplus H^J_-(\mathfrak{g})$, and*
- *$\bar{\Psi}^\pm : H_J^\pm(\mathfrak{g}) \times H^J_\pm(\mathfrak{g}) \to \mathbb{R}$ is non-degenerate.*

3.3. **Cohomological characterization of tamed J.** In this subsection we are going to prove Theorem 0.4.

For a linear subspace $V \subset \Lambda^2(\mathfrak{g}^*)$, let $n(V)$ be the maximal dimension of an isotropic subspace with respect to Φ_ζ, and $b^+(V)$ be the maximal dimension of a positive definite subspace with respect to Φ_ζ. We use $b^+(\mathfrak{g})$ for $b^+(\mathcal{Z})$. We have the following

Lemma 3.6. *Let $\mathfrak{g}$ be a $4-$dimensional unimodular Lie algebra.*
(i) $\dim \mathcal{Z} = b^+(\mathfrak{g}) + 3$ *and* $n(\mathcal{Z}) = 3 - b^+(\mathfrak{g})$.
(ii) $\dim \mathcal{B} = n(\mathcal{B}) = 3 - b^+(\mathfrak{g})$.
(iii) $\dim(\Lambda_J^+(\mathfrak{g}^*) \cap \mathcal{B}) \leq 1$.
(iv) $\dim(\Lambda_J^+(\mathfrak{g}^*) \cap \mathcal{Z}) = b^+(\mathfrak{g}) + 1$.
(v) $\mathcal{B} \cap \Lambda_J^-(\mathfrak{g}^*) = \{0\}$.
(vi) $b^+(\mathfrak{g}) - 1 \leq h_J^- = \dim(\Lambda_J^-(\mathfrak{g}^*) \cap \mathcal{Z}) \leq b^+(\mathfrak{g})$.

Proof. (i)+(ii) It follows from the third part of Lemma 3.2 that $(\mathcal{Z}, \mathcal{B})$ is a $\Phi_\zeta-$complementary pair. In particular, $\mathcal{B}$ is isotropic. Moreover,

$$\dim \mathcal{Z} + \dim \mathcal{B} = 6 \quad \text{and} \quad \dim \mathcal{Z} - \dim \mathcal{B} = 2b^+(\mathfrak{g}).$$

(iii) We just need to exclude the case that $\mathcal{B} \subset \Lambda_J^+(\mathfrak{g}^*)$ when $b^+(\mathfrak{g}) = 1$. This is impossible because

$$n(\mathcal{B}) = 2 \quad \text{and} \quad n(\Lambda_J^+(\mathfrak{g}^*)) = 1.$$

To see that $n(\Lambda_J^+(\mathfrak{g}^*)) = 1$, notice that any isotropic subspace is at least codimension 3 since $b^+(\Lambda_J^+(\mathfrak{g}^*)) = 3$.

(iv) Consider the intersection $\Lambda_J^+(\mathfrak{g}^*) \cap \mathcal{Z}$. First of all, by (i),

$$b^+(\mathfrak{g}) + 1 \leq \dim(\Lambda_J^+(\mathfrak{g}^*) \cap \mathcal{Z}) \leq \dim(\Lambda_J^+(\mathfrak{g}^*) = 4.$$

So $\dim(\Lambda_J^+(\mathfrak{g}^*) \cap \mathcal{Z}) = 4$ when $b^+(\mathfrak{g}) = 3$.

Suppose $b^+(\mathfrak{g}) = 2$. Then $\Lambda_J^+(\mathfrak{g}^*)$ cannot be contained in $\mathcal{Z}$ since

$$b^+(\Lambda_J^+(\mathfrak{g}^*)) = 3 \quad \text{and} \quad b^+(\mathcal{Z}) = 2.$$

Thus $\Lambda_J^+(\mathfrak{g}^*) \cap \mathcal{Z}$ always has dimension 3.

Suppose $b^+(\mathfrak{g}) = 1$. Let L^+ be a 3-dimensional positive definite subspace of $\Lambda_J^+(\mathfrak{g}^*)$. We claim that $\Lambda_J^+(\mathfrak{g}^*) \cap \mathcal{Z}$ cannot have dimension 3 or 4. This is because for any 3-dimensional subspace $L \subset \Lambda_J^+(\mathfrak{g}^*)$,

$$b^+(\mathcal{Z}) = 1 \quad \text{but} \quad b^+(\Lambda_J^+(\mathfrak{g}^*)) \geq b^+(L \cap L^+) = 2.$$

(v) This is clear since $\mathcal{B}$ is isotropic and $\Lambda_J^-(\mathfrak{g}^*)$ is positive definite.

(vi) By (v) and (7) we have $h_J^- = \dim(\Lambda_J^-(\mathfrak{g}^*) \cap \mathcal{Z})$. Since $\Lambda_J^-(\mathfrak{g}^*)$ is positive definite we have $\dim(\Lambda_J^-(\mathfrak{g}^*) \cap \mathcal{Z}) \leq b^+(\mathfrak{g})$. Since $\dim(\Lambda_J^-(\mathfrak{g}^*)) = 2$, by (i), we have $b^+(\mathfrak{g}) - 1 \leq \dim(\Lambda_J^-(\mathfrak{g}^*) \cap \mathcal{Z})$ □

We now characterize tameness in terms of $h_J^\pm$, from which Theorem 0.4 follows.

Theorem 3.7. *Let $\mathfrak{g}$ be a $4-$dimensional unimodular Lie algebra. The following are equivalent.*
0) J is tamed;

1) $\mathcal{B}\cap\Lambda_J^+(\mathfrak{g}^*)=\{0\}$*;*

2) $h_J^+=b^+(\mathfrak{g})+1$*;*

3) $h_J^-=b^+(\mathfrak{g})-1$*;*

4) $H_J^+(\mathfrak{g})$ *has signature* $(1,b^+(\mathfrak{g}))$*.*

Proof. 0)=1) By the first part of Lemma 2.2, we obtain that $G_\zeta(\Lambda_J^+(\mathfrak{g}^*))=\Lambda_J^+(\mathfrak{g})$ and by the second part of Lemma 3.2, that $G_\eta(B)=\mathcal{B}$. Therefore, $B\cap\Lambda_J^+(\mathfrak{g})=\{0\}$ if and only if $\mathcal{B}\cap\Lambda_J^+(\mathfrak{g}^*)=\{0\}$. By Theorem 2.5, J is tamed if and only if $B\cap\Lambda_J^+(\mathfrak{g})=\{0\}$. Hence 0)=1) is proved.

0)=2) This is about the intersection $\Lambda_J^+(\mathfrak{g}^*)\cap\mathcal{Z}$. It cannot contain $\mathcal{B}$ if J is tamed. Thus in this case, by (7), we have $h_J^+=\Lambda_J^+(\mathfrak{g}^*)\cap\mathcal{Z}$, which is equal to $b^+(\mathfrak{g})+1$ by the fourth part of Lemma 3.6.

Suppose J is not tamed, then $\Lambda_J^+(\mathfrak{g}^*)\cap\mathcal{B}$ is 1 dimensional by the third part of Lemma 3.6. Thus $h_J^+=b^+(\mathfrak{g})$.

2)=3) Since $h_J^-=b^2-h_J^+$ by Theorem 3.3, we have the required inequality $h_J^-=b^+(\mathfrak{g})-1$.

4)=2) It suffices to show that 2)⇒4). This is true because 2)⇒3), and 3)⇒ $b^+(H_J^+(\mathfrak{g}))=1$.

The proof of Theorem 3.7 is complete. □

Corollary 3.8. *Let* $\mathfrak{g}$ *be a* $4-$*dimensional unimodular Lie algebra.*

- *When* $b^+(\mathfrak{g})=3$*, every* J *is tamed and complex, and has* $h_J^-=2$*.*
- *When* $b^+(\mathfrak{g})=2$*,* J *is either tamed or complex, corresponding to* $h_J^-=1$ *or* $\Lambda_J^-(\mathfrak{g}^*)\subset\mathcal{Z}$ *respectively.*
- *When* $b^+(\mathfrak{g})=1$*,* J *is either tamed or* $\Lambda_J^-(\mathfrak{g}^*)\cap\mathcal{Z}\neq\{0\}$*.*

Proof. For the three bullets, only the second one requires a proof. In this case, $b^+(\mathfrak{g})=2$, J is tamed if and only if $h_J^-=1$. If J is not tamed then $h_J^-=2$, namely, $\Lambda_J^-(\mathfrak{g}^*)\subset\mathcal{Z}$. By Lemma 2.7 such a J is integrable. □

Together with Proposition 2.6, Corollary 3.8 can be used to show that there are non-tamed almost complex structure on any $4-$dimensional unimodular Lie algebra with $b^+(\mathfrak{g})\leq 2$. We will be content to illustrate this point with examples in 3.5.

3.4. **Tamed and almost Kähler cone.** In this subsection we prove Theorem 0.6. We start with a description of the cones $\mathcal{K}_J^t$ and $\mathcal{K}_J^c$ in terms of HPC_J in arbitrary dimension.

Proposition 3.9. *Let* J *be an almost complex structure on a Lie algebra* $\mathfrak{g}$*.*

If J *is tamed, then under the (non-degenerate) pairing between* $H^2(M;\mathbb{R})$ *and* $H_2(M;\mathbb{R})$*, the* $J-$*tamed cone* $\mathcal{K}_J^t\subset H^2(M;\mathbb{R})$ *is the interior of the dual cone of* $HPC_J\subset H_2(M;\mathbb{R})$*.*

If J *is almost Kähler, then under the pairing between* H_J^+ *and* H_+^J*, the* $J-$*compatible cone* $\mathcal{K}_J^c\subset H_J^+(\mathfrak{g})$ *is the interior of the dual cone of* $HPC_J\subset H_+^J(M)$*.*

Proof. The proofs are similar, so we just prove the slightly harder second statement. First of all, under the (possibly degenerate) pairing between $H^+_J(\mathfrak{g})$ and $H^J_+(\mathfrak{g})$, $\mathcal{K}^c_J$ is contained in the interior of the dual cone of HPC_J.

It remains to prove that if $e \in H^+_J(\mathfrak{g})$ is positive on $HPC_J\backslash\{0\}$, then it is represented by a $J-$compatible form.

Consider the isomorphism σ in Lemma 1.6 (v), and the element $\sigma(e)$ in $\hom(\frac{\pi_+ Z}{\pi_+ B}, \mathbb{R})$, which pulls back to a functional L on $\pi_+ Z$ vanishing on $\pi_+ B$. Denote the kernel hyperplane of L in $\pi_+ Z$ also by L. By our choice of e, as subsets of $\Lambda^+_J(\mathfrak{g})$, L and $PC_J\backslash\{0\}$ are disjoint.

By Lemma 1.10, we get a hyperplane $\mathcal{L}$ in $\Lambda^+_J(\mathfrak{g})$ containing L and disjoint from $PC_J\backslash\{0\}$. The hyperplane $\mathcal{L}$ determines a functional α on $\Lambda^+_J(\mathfrak{g})$, vanishing on $L \supset \pi_+ B$ and being positive on $PC_J\backslash\{0\}$.

By Lemma 1.11, α is a $J-$compatible symplectic form. Moreover, by construction we have $[\alpha] = e$. □

Finally, we restate and prove Theorem 0.6.

Theorem 3.10. *For any tamed structure J on a $4-$dimensional unimodular Lie algebra, the tamed cone $\mathcal{K}^t_J$ is $\mathcal{K}^c_J + H^-_J$, and the compatible cone $\mathcal{K}^c_J$ is a connected component of*

$$\mathcal{P}_J = \{e \in H^+_J(\mathfrak{g}) \mid e^2 > 0\},$$

where $e^2 = \bar{\Phi}_\zeta(e, e)$.

Proof. The dimension of the compatible cone is the dimension of $H^+_J(\mathfrak{g})$. Recall we define HPC_J to be the cone in $H^J_+(\mathfrak{g})$ generated by classes of closed positive vectors, $HPC_J = \{[u] \mid u \in PC_J \cap Z\}$. By Proposition 3.9, the compatible cone is the $H^+_J - H^J_+$ dual of the cone HPC_J, and the tamed cone is the $H^2 - H_2$ dual of the cone HPC_J. Furthermore, the tamed cone is the sum of the compatible cone and $H^-_J(\mathfrak{g})$.

Now we determine the $H^+_J - H^J_+$ dual of HPC_J. When J is tamed, $H^+_J(\mathfrak{g})$ has signature $(1, b^+(\mathfrak{g}))$ by part 4) of Theorem 3.7. Any positive vector $u \in PC_J$ satisfies $\Phi_\eta(u, u) \geq 0$ by Lemma 2.2 (iv). By the light cone lemma and Lemma 2.2 (i), we have the $H^+_J - H^J_+$ dual of HPC_J is a connected component of $\mathcal{P}_J$. □

3.5. **Examples.** If J is a tamed almost complex structure on $\mathfrak{g}$, then $\mathfrak{g}$ is necessarily symplectic. According to [4, Theorem 9], a four dimensional symplectic Lie algebra is solvable. For the classification of the four dimensional solvable Lie algebras see e.g.[1, Theorem 1.5] (where product structures are investigated) or [24, Theorem 4.1].

The unimodular symplectic Lie algebras are classified in [27]. According to that list, in each case we can find a basis $\{f_1, \dots, f_4\}$ of $\mathfrak{g}$, such that

$$\begin{array}{lll}
0) & \mathbb{R}^4, & [f_i, f_j] = 0\,, \quad i, j = 1, \dots, 4 \\
1) & \mathfrak{nil}^3 \times \mathbb{R}, & [f_1, f_3] = f_2, \\
2) & \mathfrak{nil}^4, & [f_4, f_2] = f_1, \quad [f_4, f_3] = f_2, \\
3) & \mathfrak{sol}^3 \times \mathbb{R}, & [f_4, f_1] = f_1, \quad [f_2, f_4] = f_2, \\
4) & \mathfrak{r}'_{3,0} \times \mathbb{R}, & [f_1, f_3] = f_2, \quad [f_2, f_1] = f_3.
\end{array}$$

Denoting by $\{f^1, \dots, f^4\}$ the dual basis of $\{f_1, \dots, f_4\}$, we set $f^{ij} := f^i \wedge f^j$ and so on. It follows from Proposition 2.6 and the last two bullets of Corollary 3.8 that there

are non-tamed J when $b^+(\mathfrak{g}) = 1, 2$. Applying the formula (17) and Corollary 3.8 we will produce explicitly a 2-parameter and a 1-parameter family respectively of non-tamed almost complex structures on Lie algebras $\mathfrak{nil}^3 \times \mathbb{R}$ and $\mathfrak{nil}^4$, having $b^+(\mathfrak{g}) = 2$ and $b^+(\mathfrak{g}) = 1$ respectively.

1) On $\mathfrak{nil}^3 \times \mathbb{R}$, consider the 2-parameter family $\{J_{a,b}\}$ of almost complex structures given, with respect to the frame $\{f_1, \dots, f_4\}$, by the following matrix

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ a & 0 & -b & -1 \\ 1 & 0 & 0 & 0 \\ -b & 1 & -a & 0 \end{pmatrix},$$

where a, b are real parameters such that $a^2 + b^2 \neq 0$. A direct computation, by using formula (17), shows that

$$\begin{aligned} \Lambda^-_{J_{a,b}}(\mathfrak{g}^*) &= \mathrm{Span}\{a(f^{34} - f^{12}) - b(f^{23} - f^{14}) + (a^2 + b^2)f^{13}, a(f^{23} - f^{14}) + b(f^{34} - f^{12})\} \\ &\subset \mathcal{Z} = \mathrm{Span}\{f^{12}, f^{34}, f^{14}, f^{23}, f^{13}\}. \end{aligned}$$

Since $b^+(\mathfrak{g}^*) = 2$, according to the second bullet of Corollary 3.8, J cannot be tamed by any symplectic form. Furthermore, these almost complex structures must be integrable.

2) On $\mathfrak{nil}^4$, let $\{J_t\}$, for $|t| < 1$, be represented, with respect to the frame $\{f_1, \dots, f_4\}$, by the following matrix,

$$\frac{1}{1+t^2}\begin{pmatrix} 0 & t^2-1 & 2t & 0 \\ 1-t^2 & 0 & 0 & -2t \\ -2t & 0 & 0 & t^2-1 \\ 0 & 2t & 1-t^2 & 0 \end{pmatrix}.$$

Then, by formula (17), we obtain, after a straightforward computation, that

$$\Lambda^-_{J_t}(\mathfrak{g}^*) \ni f^{23} + f^{14} \in \mathcal{Z} = \mathrm{Span}\{f^{14}, f^{23}, f^{24}, f^{34}\}.$$

Since $b^+(\mathfrak{g}^*) = 1$, according to the third bullet of Corollary 3.8, $\{J_t\}$ cannot be tamed.

We next offer a fun verification of Theorem 3.10 for an explicit J on $\mathfrak{g} = \mathfrak{nil}^3 \times \mathbb{R}$. A direct computation shows that the pair (J, ω) defined by

$$\begin{gathered} Jf_1 = f_2\,, \quad Jf_2 = -f_1\,, \quad Jf_3 = f_4\,, \quad Jf_4 = -f_3 \\ \omega = f^{12} + f^{34}\,, \end{gathered} \tag{27}$$

gives rise to an almost Kähler structure on $\mathfrak{g}$ with $\zeta = f_1 \wedge Jf_1 \wedge f_3 \wedge Jf_3$.
We immediately get that

$$\begin{aligned} H^+_J(\mathfrak{g}) &= \mathrm{Span}_{\mathbb{R}}\{[f^{12}], [f^{34}], [f^{14} - f^{23}]\}\,, \\ H^-_J(\mathfrak{g}) &= \mathrm{Span}_{\mathbb{R}}\{[f^{14} + f^{23}]\}. \end{aligned}$$

Pick the connected component of $\mathcal{P}_J$ containing $[\omega]$, and denote it by T. Any class in T is represented by a form $\alpha = af^{12} + bf^{34} + c\left(f^{14} - f^{23}\right) \in \mathcal{Z}^+_J$ with $\Phi_\zeta(\alpha, \alpha) > 0$ and $\Phi_\zeta(\alpha, \omega) > 0$, which means

$$ab - c^2 > 0\,, \quad a > 0\,, \quad b > 0.$$

To show that α is $J-$compatible, we need to check that α is positive, i.e., $\alpha(v \wedge Jv) > 0$, for every $v \neq 0$. If

$$v = Af_1 + Bf_2 + Cf_3 + Df_4,$$

then

$$Jv = -Bf_1 + Af_2 - Df_3 + Cf_4.$$

It follows that

$$\alpha(v, Jv) = \left(af^{12} + bf^{34} + c(f^{14} - f^{23})\right)(v, Jv) > 0$$

if and only if

$$a(A^2 + B^2) + b(C^2 + D^2) > -2c(AC + BD). \tag{28}$$

The last condition (28) is equivalent to

$$a^2(A^2 + B^2)^2 + b^2(C^2 + D^2)^2 + 2ab(A^2 + B^2)(C^2 + D^2) - 4c^2(AC + BD)^2 > 0. \tag{29}$$

Now,

$$\begin{aligned} 4c^2(AC + BD)^2 &= 4c^2(A^2C^2 + B^2D^2 + 2ABCD) \leq 4c^2(A^2 + B^2)(C^2 + D^2) \\ &< 4ab(A^2 + B^2)(C^2 + D^2) \\ &\leq a^2(A^2 + B^2)^2 + b^2(C^2 + D^2)^2 + 2ab(A^2 + B^2)(C^2 + D^2), \end{aligned}$$

i.e. (29) is proved. Hence, the compatible cone of the almost Kähler J on the Lie algebra $\mathfrak{nil}^3 \times \mathbb{R}$ is the connected component T of $\mathcal{P}_J$.

4. Compact quotients of 4-dimensional Lie groups

The main results in this paper are inspired by some recent advances on the geometry of compact 4−dimensional almost complex manifolds, which we offer a quick review.

For a $2n$-dimensional compact almost complex manifold (M, J), J acts on the bundle of real 2-forms Λ^2 as an involution, by $\alpha(\cdot, \cdot) \mapsto \alpha(J\cdot, J\cdot)$, thus we have the splitting at the bundle level $\Lambda^2 = \Lambda_J^+ \oplus \Lambda_J^-$, and the splitting at the form level $\Lambda^2 = \Lambda_J^+ \oplus \Lambda_J^-$. The corresponding cohomology groups $H_J^+(M)$ and $H_J^-(M)$ were introduced in [22].
The motivation of introducing these groups is to better understand the question of Donaldson, mentioned in the introduction and rephrased here in terms of the tamed and almost Kähler cones

$$\mathcal{K}_J^t = \left\{[\omega] \in H^2(M) \mid \omega \text{ is } J\text{-tamed}\right\} \quad \text{and} \quad \mathcal{K}_J^c = \left\{[\omega] \in H_J^+(M) \mid \omega \text{ is } J\text{-compatible}\right\}.$$

Question *Let (X, J) be a compact 4-dimensional almost complex manifold. Then, is it true that*

$$\mathcal{K}_J^t \neq \emptyset \iff \mathcal{K}_J^c \neq \emptyset\,?$$

In [9], T. Drăghici, the first author and W. Zhang proved that on any compact almost complex 4-dimensional manifold (M, J), the 2-nd de Rham cohomology group decomposes as

$$H^2(M; \mathbb{R}) = H_J^+(M) \oplus H_J^-(M).$$

These groups can be viewed as a generalization of Dolbeault groups to the non-integrable case. Indeed, in [9, Prop.2.7] (see also [10]) it is proved that, if J is a complex structure on

a 4-dimensional compact manifold, then $H^{\pm}_J(M)$ are identified with real Dolbeault groups, namely,

$$H^+_J(M) \simeq H^{1,1}_{\overline{\partial}}(M) \cap H^2(M;\mathbb{R}), \qquad H^-_J(M) \simeq \left(H^{2,0}_{\overline{\partial}}(M) \oplus H^{0,2}_{\overline{\partial}}(M)\right) \cap H^2(M;\mathbb{R}) \tag{30}$$

The analogues of Proposition 3.9 and Theorem 3.10 were first established in [22]. For higher dimensional results see also [2].

We end this paper with some applications of the main results to homogeneous almost complex structures.

4.1. **Nomizu-Hattori type theorem for $H^{\pm}_J(M)$.** The well known Nomizu-Hattori Theorem (see [26], [18]) states that if $\Gamma\backslash G$ is a compact solvmanifold of *completely solvable type*, i.e., the adjoint representation of $\mathfrak{g}$ has real eigenvalues, then $H^*(\Gamma\backslash G) \cong H^*(\mathfrak{g})$.

We prove a Nomizu-Hattori type theorem for $H^{\pm}_J(M)$, where M is a 4-dimensional compact quotient of a simply-connected Lie group G, more precisely

Theorem 4.1. *Let $M = \Gamma\backslash G$ be a compact quotient of a simply-connected 4-dimensional real Lie group G by a uniform discrete subgroup $\Gamma \subset G$ and let J be an almost complex structure on the Lie algebra $\mathfrak{g}$ of G. Assume that $H^2(\mathfrak{g}) \cong H^2(M)$. Then*

$$H^+_J(M) = H^+_J(\mathfrak{g}), \quad H^-_J(M) = H^-_J(\mathfrak{g}).$$

Proof. In view of a result of Milnor (see [25, Lemma 6.2]), the existence of a uniform discrete subgroup of G implies that the Lie algebra $\mathfrak{g}$ of G has to be unimodular. Let J be the induced left-invariant almost complex structure on $M = \Gamma\backslash G$. In view of the assumption $H^2(\mathfrak{g}) \cong H^2(M)$, of the fact that J is $\mathcal{C}^\infty$ pure and full [9, Theorem 2.3] and of Theorem 3.3, we have that

$$H^2(\mathfrak{g}) = H^+_J(\mathfrak{g}) \oplus H^-_J(\mathfrak{g}) \overset{\cong}{\hookrightarrow} H^2(M;\mathbb{R}) = H^+_J(M) \oplus H^-_J(M)$$

Noticing that there are natural homomorphisms,

$$H^+_J(\mathfrak{g}) \longrightarrow H^+_J(M), \quad H^-_J(\mathfrak{g}) \longrightarrow H^-_J(M),$$

the conclusion follows. $\square$

Remark 4.2. *For the Dolbeault cohomology, in* [5] *(see also* [6]*) a Nomizu-Hattori theorem is proved: more precisely, if $M = \Gamma\backslash G$ is a compact nilmanifold endowed with an* Abelian *left-invariant complex structure J, namely, J is a complex structure on $\mathfrak{g}$ such that $[Ju, Jv] = [u, v]$, for every $u, v \in \mathfrak{g}$, then $H^{p,q}_{\overline{\partial}}(M) \simeq H^{p,q}_{\overline{\partial}}(\mathfrak{g}_{\mathbb{C}})$.*
Let J be a complex structure on a 4-dimensional Lie algebra $\mathfrak{g}$. Then arguing as in [9, Prop.2.7]*, it can be showed that the same decompositions* (30) *hold, for real Dolbeault groups of $\mathfrak{g}$. Therefore, in dimension four, under the weaker assumption that G is completely solvable, a consequence of Theorem 4.1 is a Nomizu-Hattori theorem for real Dolbeault groups, namely,*

$$H^{1,1}_{\overline{\partial}}(M) \cap H^2(M;\mathbb{R}) \simeq H^{1,1}_{\overline{\partial}}(\mathfrak{g}) \cap H^2(\mathfrak{g})$$

$$\left(H^{2,0}_{\overline{\partial}}(M) \oplus H^{0,2}_{\overline{\partial}}(M)\right) \cap H^2(M;\mathbb{R}) \simeq \left(H^{2,0}_{\overline{\partial}}(\mathfrak{g}) \oplus H^{0,2}_{\overline{\partial}}(\mathfrak{g})\right) \cap H^2(\mathfrak{g}).$$

4.2. **Tamed and almost Kähler structures on quotients.** Finally, we note that there are analogues of Theorems 0.2, 0.4, and 0.6.

Theorem 4.3. *Let $M = \Gamma\backslash G$ be a compact quotient of a 4-dimensional simply-connected Lie group G, where $\Gamma \subset G$ is a uniform discrete subgroup. Let J be a left-invariant almost complex structure on M. Then the following are equivalent:*

0) J is tamed by a symplectic form ω on M (not necessarily left-invariant).
1) J is tamed by an left-invariant symplectic form ω on M.
2) There are no non-trivial J-invariant exact 2-forms on M.
3) M carries a left-invariant symplectic form compatible with J.
4) M carries a symplectic form compatible with J (not necessarily left-invariant).
5) $h_J^- = b^+(M) - 1$.

Furthermore, if J is tamed, the $J-$compatible cone is a connected component of $\mathcal{P}_J = \{e \in H_J^+(M) \mid e^2 > 0\}$.

As already recalled, the Lie algebra $\mathfrak{g}$ of G is unimodular. Assume that ω is a non left-invariant symplectic structure on M taming J (compatible with J). Then by the same process as in [14], we may construct a left-invariant symplectic structure $\tilde{\omega}$ on M, taming J (compatible with J).

In view of these facts, all the statements follow from the corresponding results for 4-dimensional unimodular Lie algebras, specifically, Theorems 3.7, 2.5, 3.10.

It is amusing to notice that, for a compact 4-manifold, tameness of an integrable almost complex structure ([17], see also [11]) and tameness of a homogeneous almost complex structure can be both characterized by $h_J^- = b^+ - 1$.

We note that there is an alternative argument for the equivalence between 1) and 3) in [11]. We also note that the Kähler cone is determined in terms of the curve cone by Lamari, Buchdahl in [20] and [3] for a Kähler surface, by Demailly and Paun in [7] for an arbitrary Kähler manifold, which generalizes the famous Nakai-Moizheson ampleness criterion in projective geometry. In the almost Kähler case in dimension 4, such a generalization is contained in Taubes' [30] for a generic J when $b^+ = 1$, and for several geometrically interesting families of almost complex structures on rational manifolds in [23].

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